Small universal Turing machines

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Abstract

Let UTM(m,n) be the class of universal Turing machine with m states and n symbols. Universal Turing machines are proved to exist in the following classes: UTM(24,2), UTM(10,3), UTM(7,4), UTM(5,5), UTM(4,6), UTM(3,10) and UTM(2,18).

1. Introduction

In 1956, Shannon [17] posed the problem of the construction of the simplest universal Turing machine. He was considering ordinary deterministic Turing machine, with a two-way infinite tape and one head. He proposed to measure the complexity of such a Turing machine by the number of commands of this machine, that is the product \( mn \) of the number \( m \) of states by the number \( n \) of tape symbols. It is also possible to consider the number of commands really used by the machine.

Let UTM(m,n) denote the class of universal Turing machines with \( m \) states and \( n \) symbols. Various definitions of universal Turing machine, and the one we choose, will be discussed in Section 2. Pavlotskaya proved that the classes UTM(3,2) [10] and UTM(2,3) (unpublished) are empty. Using another method, Diekert and Kudlek [3], and Kudlek [5] proved that UTM(2,2) is empty.

The main result of this paper is the following theorem:

**Main Theorem.** There are universal Turing machines in the following seven classes: UTM(24,2), UTM(10,3), UTM(7,4), UTM(5,5), UTM(4,6), UTM(3,10) and UTM(2,18).

This theorem and the results of Pavlotskaya leave 51 classes UTM(m,n) with an unsettled emptiness problem.

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Minsky [8] constructed a Turing machine in UTM(7,4), by simulating tag-systems. We use the same method, but the machines in UTM(5,5) and UTM(4,6) are simulating particular classes of tag-systems.

The machines in UTM(24,2), UTM(7,4), UTM(5,5) and UTM(4,6) were already presented in [13]. The machines in UTM(11,3), UTM(3,10) and UTM(2,21) given in [13] are now replaced by machines in UTM(10,3) [14], UTM(3,10) [15] (with 27 commands instead of 28 commands for the one in [13]) and UTM(2,18) [16].

Minsky’s machine in UTM(7,4) has an unsolvable halting problem, but has a defect: it damages the output, and, therefore, it cannot compute all partial recursive functions or simulate all Turing machines. Our machine in UTM(7,4) does not have such a defect. Robinson [12] also noticed this defect and gave a machine in UTM(7,4) where the output is preserved and is given immediately to the right of the head of the Turing machine. Robinson considered the number of commands really used in the program of the Turing machine. His machine in UTM(7,4) uses 27 commands, whereas ours uses 26 commands. Note that our Turing machine in UTM(5,5) uses 23 commands, and the one in UTM(4,6) uses 22 commands, which is the least known number of commands for a universal Turing machine. Robinson checked and analysed the machines presented in [13] and the machine in UTM(2,18). Margenstern [7] considers Shannon’s problem for non-erasing Turing machines.

The paper is structured as follows. In Section 2, we present various definitions of universality for Turing machines among which two equivalent definitions are retained. The general principles of construction of universal Turing machines simulating tag-systems are presented in Section 3. In Sections 4 to 10, seven universal Turing machines, belonging to the classes mentioned in the Main Theorem, are defined and analysed.

2. Equivalent definitions for universal Turing machines

We deal with the ordinary notion of deterministic Turing machine (TM) with one-dimensional tape and one head and those of configuration of TM and tag-system (cf. [2]). First we consider the notion of universality for Turing machines.

In what follows, denote via \( \alpha, \beta \) (with or without subscripts) configurations of a TM (tag-system or other algorithm model) and via \( \beta \perp \) the fact that the configuration \( \beta \) is final. Let \( \beta_1 \xrightarrow{M} \beta_2 \) mean that TM M moves from a configuration \( \beta_1 \) to a configuration \( \beta_2 \) by one step and we write \( \beta_1 \xrightarrow{M} \beta \) for \( \beta_1 \xrightarrow{M} \beta_2 \xrightarrow{M} \cdots \xrightarrow{M} \beta_j \).

Let \( M \) be a fixed TM. Denote via \( B \) the set of all configurations of all Turing machines and via \( B_M \) the set of configurations of \( M \). It is well-known that both \( B \) and \( B_M \) are recursive sets. Also, we define a function \( F_M \) as follows:

\[
F_M(\alpha) = \beta \quad \text{if and only if} \quad \alpha \xrightarrow{M} \beta,
\]

where \( \alpha, \beta \in B_M \) and \( \beta \perp \). We denote the domain of \( F \) and the range of \( F \) with \( \text{Def}(F) \) and \( \text{Val}(F) \), respectively.
Definition 1. Let $\psi(n,x)$ be some universal partial recursive function for all partial recursive functions of one variable. A TM $U$ is called universal (UTM), if there exists a total recursive (or simple-recursive) function $\rho(n,x)$, the coding function with $\Val(\rho) \subseteq B_U$, and a recursive function $\lambda(\alpha)$, the decoding function, defined on the set $B_U$, so that for all $n$ and $x$ the following holds:

$$\lambda(F_U(\rho(n,x))) = \psi(n,x).$$

Remark 1. Definition 1 of the UTM coincides with the definition of the UTM by Davis in [1], except for some immaterial details.

Definition 2. A TM $M$ simulates the TM $T$, if there exists a recursive function $\tilde{\rho}(\alpha)$, the coding function, defined on the set $B_T$, with $\Val(\tilde{\rho}) \subseteq B_M$ and a recursive function $\tilde{\lambda}(\alpha)$, the decoding function, defined on the set $B_M$, with $\Val(\tilde{\lambda}) \subseteq B_T$, so that for all $\alpha \in B_T$, the following holds:

$$\tilde{\lambda}(F_M(\tilde{\rho}(\alpha))) = F_T(\alpha).$$

Remark 2. Definition 2 is based on the definition of the simulation of one abstract computing machine by another according to Herman [4].

Let $T_n$ be a TM with Gödel number $n$.

Definition 3. A TM $U$ is called a universal Turing machine if one can simulate each TM $M$ and one can effectively get coding and decoding functions from the program of the TM $M$, i.e. there exists a recursive function $\tilde{\rho}(n,\alpha)$ defined on $N \times B$ with $\Val(\tilde{\rho}) \subseteq B_U$ and a recursive function $\tilde{\lambda}(n,\alpha)$ defined on $N \times B_U$ with $\Val(\tilde{\lambda}) \subseteq B$ so that for all $< n, \alpha > \in N \times B$ the following holds:

$$\tilde{\lambda}(n,F_U(\tilde{\rho}(n,\alpha))) = F_{T_n}(\alpha).$$ (1)

Remark 3. It is easy to show that the decoding function can be of one variable, i.e. $\tilde{\lambda}(\alpha)$, Def($\tilde{\lambda}$) = $B_U$, Val($\tilde{\lambda}$) $\subseteq B$ and expression (1) can be rewritten as follows:

$$\tilde{\lambda}(F_U(\tilde{\rho}(n,\alpha))) = F_{T_n}(\alpha).$$ (2)

Theorem. Definitions 1 and 3 of the universality for Turing machines are equivalent.

A proof of the Theorem above is grounded on the following three lemmas.

Lemma 2.1. Let $M$ be an UTM with respect to Definition 1. Then $M$ calculates arbitrary binary partial recursive function.

Proof. Obvious. $\square$
Let $T_n$ be the TM with Gödel number $n$ and $G(x)$ be a Gödel enumeration of the set $B$ of all configuration of all TMs.

With each TM $M$, we associate a number function $\varphi_M$ as follows:

$$\varphi_M(x) = y \iff x \in G^{-1}(B_M) \land y \in G^{-1}(B_M) \land F_M(G(x)) = G(y).$$  \hfill (3)

The function $\varphi_M(x)$ is a partial recursive function, because $B_M$ is a recursive set.

Let $F_M(x)$ be not defined, if $x \notin B_M$. It follows from (3) that

$$(\forall x \in \mathbb{N})[\varphi_M(x) = G^{-1} \circ F_M \circ G(x)].$$  \hfill (4)

**Lemma 2.2.** Each UTM with respect to Definition 1 is an UTM with respect to Definition 3.

**Proof.** It is obvious that $t(n,x) = \varphi_{T_n}(x)$ is a partial recursive.

According to Lemma 2.1, there are recursive $\lambda(x)$ and $\rho(n,x)$, such that for all $<n,x>$ the following holds:

$$\lambda \circ F_M \circ \rho(n,x) = \varphi_{T_n}(x).$$ \hfill (5)

From (4) we have

$$(\forall n,x \in \mathbb{N})[\varphi_{T_n}(x) = G^{-1} \circ F_{T_n} \circ G(x)].$$ \hfill (6)

Then we have from (5) and (6)

$$\lambda \circ F_M \circ \rho(n,x) = G^{-1} \circ F_{T_n} \circ G(x)$$

and

$$G \circ \lambda \circ F_M \circ \rho(n,x) = F_{T_n} \circ G(x).$$

Let $x = G^{-1}(\alpha)$, where $\alpha \in B$. Then

$$(\forall \alpha \in B)[G \circ \lambda \circ F_M \circ \rho(n,G^{-1}(\alpha)) = F_{T_n}(\alpha)].$$

Let $\tilde{\lambda}(\beta) = G \circ \lambda(\beta)$ for $\beta \in B_M$ and $\tilde{\rho}(n,\alpha) = \rho(n,G^{-1}(\alpha))$, where $\alpha \in B$.

Then $\tilde{\lambda} \circ F_M \circ \tilde{\rho}(n,\alpha) = F_{T_n}(\alpha)$ for all $\alpha \in B$ and TM $M$ is an UTM according to Definition 3. \qed

**Lemma 2.3.** Each UTM according to Definition 3 is an UTM according to Definition 1.

**Proof.** Let TM $M$ be universal according to Definition 3, then according to Remark 3, there exist recursive functions $\tilde{\lambda}(\alpha)$ and $\tilde{\rho}(n,\alpha)$, such that for all $<n,\alpha> \in \mathbb{N} \times B$ the following holds:

$$\tilde{\lambda} \circ F_M \circ \tilde{\rho}(n,\alpha) = F_{T_n}(\alpha).$$
Let $T_{n_0}$ be some UTM according to Definition 1. Then, in particular,
\[ \tilde{\lambda} \circ F_M \circ \tilde{\rho}(n_0, \alpha) = F_{T_{n_0}}(\alpha), \]  
and according to Definition 1, there exist recursive $\lambda(\alpha)$ and $\rho(n, x)$ such that for all $n, x$, the following holds:
\[ \lambda \circ F_{T_{n_0}} \circ \rho(n, x) - \psi(n, x), \]  
where $\psi(n, x)$ is some universal partial recursive function for all unary partial recursive functions.

From (7) and (8),
\[ \lambda \circ \tilde{\lambda} \circ F_M \circ \tilde{\rho}(n_0, \rho(n, x)) = \lambda \circ F_{T_{n_0}} \circ \rho(n, x) = \psi(n, x). \]

Let be $\lambda_1(\alpha) = \lambda \circ \tilde{\lambda}(\alpha)$ and $\rho_1(n, x) = \tilde{\rho}(n_0, \rho(n, x))$. Then
\[ \lambda_1 \circ F_M \circ \rho_1(n, x) = \psi(n, x). \]

**Definition 4** (Maltsev [6]). A TM $T$ is called universal, if Def($F_T$) is a creative set.

**Remark 4.** Maltsev's definition of the universality for TM is, in fact, wider than our two definitions above, because the TM $T$ can have a creative Def($F_T$) and calculate only a constant function.

3. Preliminaries: How to construct a universal Turing machine

The universal Turing machines we define in the following sections simulate tag-systems. For positive integer $m$ and alphabet $A = \{a_1, \ldots, a_n, a_{n+1}\}$, a $m$-tag-system on $A$ transforms word $\beta$ on $A$ as follows: we delete the first $m$ letters of $\beta$ and we append to the right of the result a word that depends on the first letter of $\beta$. This process is iterated until $m$ letters cannot be deleted or the first letter is $a_{n+1}$, and then stops. Formally, we have the following definitions.

**Definition 5.** A *tag-system* is a three-tuple $T = (m, A, P)$, where $m$ is a positive integer, $A = \{a_1, \ldots, a_n, a_{n+1}\}$ is a finite alphabet, and $P$ maps $\{a_1, \ldots, a_n\}$ into the set $A^*$ of finite words (i.e. sequences of letters) on alphabet $A$ and $a_{n+1}$ to STOP.

A tag-system $T = (m, A, P)$ is called a $m$-tag-system. The words $\alpha_i = P(a_i) \in A^*$ are called the *productions* of tag-system $T$. The letter $a_{n+1}$ is the *halting symbol*. The productions are often displayed as follows:

\[
(T) \quad \begin{cases} 
  a_i \rightarrow \alpha_i, & i \in \{1, \ldots, n\} \\
  a_{n+1} \rightarrow \text{STOP}
\end{cases}
\]

A *computation* of tag-system $T = (m, A, P)$ on word $\beta \in A^*$ is a sequence $\beta = \beta_0, \beta_1, \ldots$ of words on $A$ such that, for all nonnegative integer $k$, $\beta_k$ is transformed into
\( \beta_{k+1} \) by deleting the first \( m \) letters of \( \beta_k \) and appending word \( a_i \) to the result if the first letter of \( \beta_k \) is \( a_i \). The computation stops in \( k \) steps if the length of \( \beta_k \) is less than \( m \) or the first letter of \( \beta_k \) is \( a_{n+1} \).

**Example.** The 2-tag-system \( T_1 \) is defined by

\[
a_1 \rightarrow a_2a_1a_3, \quad a_2 \rightarrow a_1, \quad a_3 \rightarrow \text{STOP}.
\]

On the initial word \( \beta = a_2a_1a_1 \), the computation of \( T_1 \) is

\[
a_2a_1a_1 \rightarrow a_1a_1 \rightarrow a_2a_1a_3 \rightarrow a_3a_1.
\]

Minsky [8] proved the existence of a universal 2-tag-system, and, therefore, we will deal only with 2-tag-systems, which also have the following properties:

1. The computation of a tag-system stops only on a word beginning with the halting symbol \( a_{n+1} \).
2. The productions \( \alpha_i, \quad i \in \{1, \ldots, n\} \), are not empty.

Henceforth, tag-systems will be 2-tag-systems.

A universal Turing machine \( U \) simulates a tag-system as follows. Let \( T \) be a tag-system on \( A = \{a_1, \ldots, a_{n+1}\} \) with productions \( a_i \rightarrow \alpha_i \). To each letter \( a_i \in A \) is associated a positive number \( N_i \) and codes \( A_i \) and \( \tilde{A}_i \) (may be \( A_i = \tilde{A}_i \)), of the form \( u^N (= uu \ldots u, \quad N_i \text{ times}) \), where \( u \) is a string of symbols of the machine \( U \).

The codes \( A_i \) (or \( \tilde{A}_i \)) are separated by marks on the tape of \( U \).

For \( i \in \{1, \ldots, n\} \), the production \( a_i \rightarrow \alpha_i = a_{i1}a_{i2} \ldots a_{in} \) of the tag-system \( T \) is coded by

\[
P_i = A_{i1}A_{i2} \ldots A_{in}.
\]

The initial word \( \beta = a_1a_2a_3 \ldots a_w \), to be transformed by the tag-system \( T \), is coded by

\[
S = A_1A_2A_3 \ldots A_w \quad (S = \tilde{A}_1\tilde{A}_2\tilde{A}_3 \ldots \tilde{A}_w).
\]

The initial tape of the UTM is:

\[
QLP_{n+1}P_n \ldots P_0A_1A_2A_3 \ldots A_wQR,
\]

where \( Q_L \) and \( Q_R \) are respectively infinite to the left and to the right parts of the tape of the UTM and consist only of blank symbols, \( P_{n+1} \) is the code of the halting symbol \( a_{n+1} \), \( P_0 \) is the additional code consisting of several marks, and the head of the UTM is located on the left side of the code \( S \) in the state \( q_1 \) (in the case of the machine in UTM(2,18) the head of the UTM is located on the right side of the code \( P_0 \) in the state \( q_1 \)).

Let \( T \) be an arbitrary tag-system, \( S_1 \) and \( S_2 \) be the codes of the words \( \beta_1 \) and \( \beta_2 \), respectively, and \( \beta_1 \xrightarrow{T} \beta_2 \). Then the UTM \( U \) transforms:

\[
QLP_{n+1}P_n \ldots P_0S_1QR \xrightarrow{U} QLP_{n+1}P_n \ldots P_0RS_2QR
\]
The work of the UTM can be divided into three stages:

(i) On the first stage, the UTM searches the code $P_r$ corresponding to the code $A_r$ and then the UTM deletes the codes $A_r$ and $A_s$ (i.e. it deletes the mark between them).

(ii) On the second stage, the UTM writes the code $P_r$ in $Q_R$ of the tape in the reversed order.

(iii) On the third stage the UTM restores its own tape for a new cycle of modelling.

The number $N_i$ corresponding to the symbol $a_i$ ($i \in \{1, \ldots, n+1\}$) of the tag-system has the property that there are exactly $N_r$ marks at each cycle of modelling between the code $P_r$ and the code $A_r$ (in the case of the machine in UTM(2,18) there are $N_r + 1$ marks, but the additional mark in $P_0$ is deleted immediately at the beginning of the first stage). On the first stage of modelling, the head of the UTM goes through a number of marks in the part $P$ equal to the number of symbols $u$ in the code $A_r$.

After the first stage the tape of the UTM is

$$QLP_{n+1}P_n \ldots P_r+1P_rP_{r-1} \ldots P_1P_0R'A_1A_1 \ldots A_wQ_R$$

and the head of the UTM locates the mark between $A_r'$ and $A_s$. Then the UTM deletes this mark and the second stage of modelling begins.

After the second stage, the tape of the UTM is

$$QLP_{n+1}P_n \ldots P_r+1P_rP_{r-1} \ldots P_1P_0R'A_1 \ldots A_wA_1A_2 \ldots A_m, Q_R,$$

and the head of the UTM is located on the left side of $P_r''$ and the third stage of modelling begins.

After the third stage, the tape of the UTM is

$$QLP_{n+1}P_n \ldots P_1P_0RA_1 \ldots A_wA_1A_2 \ldots A_m, Q_R$$

and the head of the UTM is located on the right side of $R$.

Let $a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k$ be the symbols of the Turing machine. $a_1a_2 \ldots a_kRb_1b_2 \ldots b_k$ means that, when the head of the UTM moves to the right the group of symbols $a_1a_2 \ldots a_k$ is changed to the group $b_1b_2 \ldots b_k$. It is analogous, when the head of the UTM moves to the left ($R$ is changed to $L$).

$Ra_1a_2 \ldots a_k(b_1b_2 \ldots b_k)L$ means that the group of symbols $a_1a_2 \ldots a_k$ makes the direction of the motion of the head of the UTM change from the right to the left and changes itself to $b_1b_2 \ldots b_k$. It is analogous, when the head of UTM moves to the left ($R$ is changed to $L$).

4. The UTM with 24 states and 2 symbols

The symbols of the machine in UTM(24,2) (see [13]) are 0 (blank symbol) and 1; and the states are $q_i$ ($i = 1, \ldots, 24$).

$$N_1 = 1, \quad N_{k+1} = N_k + m_k + 2 \quad (k \in \{1, \ldots, n\}).$$
The code of the production $x_i = a_{i1}a_{i2} \ldots a_{in_i}$, $i \in \{1, \ldots, n\}$ of the tag-system is
\[ P_i = 1010(00)^{N_m}10(00)^{N_{m-1}} \ldots 10(00)^{N_1}10, \]
where $A_j = (00)^{N_j}$, $j \in \{1, \ldots, n + 1\}$, and the pair 10 is a mark.

$P_0 = 10, \quad P_{n+1} = 1110.

The code $S$ of the initial word $\beta = a_r a_s a_t \ldots a_w$, to be transformed by the tag-system, is
\[ S = (01)^{N_r}11(01)^{N_s}11(01)^{N_t} \ldots 11(01)^{N_w}, \]
where $\tilde{A}_j = (01)^{N_j}$, $j \in \{1, \ldots, n + 1\}$, and the pair 11 is a mark.

The program of the machine in UTM(24,2):

\begin{align*}
q_{10}00Rq_5 & \quad q_{20}11Rq_1 & \quad q_{30}00Lq_4 & \quad q_{40}11Lq_{12} \\
q_{11}11Rq_2 & \quad q_{21}11Lq_3 & \quad q_{31}10Lq_2 & \quad q_{41}10Lq_9 \\
q_{50}10Rq_1 & \quad q_{60}00Lq_7 & \quad q_{70}00Lq_8 & \quad q_{80}00Lq_7 \\
q_{51}00Lq_6 & \quad q_{61}11Lq_7 & \quad q_{71}10Lq_6 & \quad q_{81}11Lq_2 \\
q_{90}00Lq_{19} & \quad q_{10}11Lq_4 & \quad q_{11}00Lq_4 & \quad q_{12}00Rq_{19} \\
q_{11}11Lq_4 & \quad q_{10}10Lq_{13} & \quad q_{11}11Lq_4 & \quad q_{12}11Lq_{14} \\
q_{13}00Rq_{10} & \quad q_{14}00Lq_{15} & \quad q_{15}00Rq_{16} & \quad q_{16}00Rq_{15} \\
q_{13}11Rq_{24} & \quad q_{14}11Lq_{11} & \quad q_{15}11Rq_{17} & \quad q_{16}11Rq_{10} \\
q_{17}00Rq_{16} & \quad q_{18}00Rq_{19} & \quad q_{19}01Lq_3 & \quad q_{20}01Rq_{18} \\
q_{17}11Rq_{21} & \quad q_{18}11Rq_{20} & \quad q_{19}11Rq_{18} & \quad q_{20}10Rq_{18} \\
q_{21}00Rq_{22} & \quad q_{22}01Lq_{10} & \quad q_{23}01Rq_{21} & \quad q_{24}00Rq_{13} \\
q_{21}11Rq_{23} & \quad q_{22}11Rq_{21} & \quad q_{23}10Rq_{21} & \quad q_{24}10Lq_3 \\
\end{align*}

(i) On the first stage of modelling:

\begin{align*}
11L10 & \quad (q_{71}00Lq_6, \ q_{61}11Lq_7), \\
01L00 & \quad (q_{70}10Lq_6, \ q_{60}00Lq_7), \\
o0L00 & \quad (q_{70}00Lq_8, \ q_{80}00Lq_7), \\
L10(11)R & \quad (q_{70}10Lq_8, \ q_{81}11Rq_2, \ q_{20}11Rq_{1}), \\
10R11 & \quad (q_{11}11Rq_2, \ q_{20}11Rq_1), \\
o0R01 & \quad (q_{10}00Rq_5, \ q_{50}01Rq_1), \\
R01(00)L & \quad (q_{10}00Rq_5, \ q_{51}00Lq_6, q_{60}00Lq_7). \\
\end{align*}

If the head of the UTM moves to the right and meets the mark 11, the first stage of modelling is finished.

$R0011(0101)L$ \quad $(q_{10}00Rq_5, \ q_{50}11Rq_1, \ q_{11}11Rq_2, q_{21}11Lq_3, \ q_{31}00Lq_2, \ q_{50}00Lq_4)$ and the second stage of modelling begins.
On the second stage of modelling:

\[ L00(01)R \begin{cases} & \text{(ii) On the second stage of modelling:} \\
01L01 & (q_{410}Lq_{9}, \ q_{900}Rq_{19}, \ q_{190}1Lq_{3}, \ q_{300}Lq_{4}), \\
11L10 & (q_{410}Lq_{9}, \ q_{911}Lq_{4}), \\
0110L0111 & (q_{410}Lq_{12}, \ q_{121}1Lq_{14}, \ q_{141}1Lq_{11}, \ q_{110}0Lq_{4}). \end{cases} \]

In this case:

\[ \begin{cases} & \text{10R11} \quad (q_{181}1Rq_{20}, \ q_{201}0Rq_{18}), \\
& \text{11R10} \quad (q_{181}1Rq_{20}, \ q_{201}0Rq_{18}), \\
& \text{01R01} \quad (q_{180}0Rq_{19}, \ q_{191}1Rq_{18}), \\
& \text{R00(01)1} \quad (q_{180}0Rq_{19}, \ q_{190}1Lq_{3}, q_{300}Lq_{4}). \end{cases} \]

\[ L0010(0011)R \begin{cases} & (q_{410}Lq_{12}, q_{121}1Lq_{14}, q_{141}00Lq_{15}, q_{151}00Rq_{16}, q_{161}00Rq_{15}, q_{151}1Rq_{17}, q_{171}1Rq_{21}) \end{cases} \]

and the UTM writes the mark 11 in \( Q_R \). In this case:

\[ \begin{cases} & \text{10R11} \quad (q_{211}1Rq_{23}, \ q_{231}0Rq_{21}), \\
& \text{01R01} \quad (q_{210}0Rq_{22}, \ q_{221}1Rq_{21}), \\
& \text{11R10} \quad (q_{211}1Rq_{23}, \ q_{231}0Rq_{21}), \\
& \text{R00(11)1} \quad (q_{210}0Rq_{22}, \ q_{221}0Lq_{10}, q_{101}0Lq_{4}). \end{cases} \]

If the head of the UTM moves to the left and meets the group 1110 = \( P_{n+1} \), then the UTM halts (\( q_{410}1Lq_{12}, q_{121}1Lq_{14}, q_{141}1Lq_{11}, q_{111}1 \)).

If the head of the UTM moves to the left and meets the group 1010, the second stage of modelling is over:

\[ L1010(1010)R \begin{cases} & (q_{410}1Lq_{12}, q_{121}1Lq_{14}, q_{141}00Lq_{15}, q_{151}1Rq_{17}, q_{171}00Rq_{16}, q_{161}1Rq_{10}, q_{101}0Rq_{13}). \end{cases} \]

On the third stage of modelling:

\[ \begin{cases} & \text{10R10} \quad (q_{131}1Rq_{24}, q_{240}0Rq_{13}), \\
& \text{01R00} \quad (q_{130}0Rq_{10}, q_{101}0Rq_{13}). \end{cases} \]

When the head of the UTM moves to the right and meets the mark 11 (in front of the code \( \bar{A}_r \)), then both the third stage and the whole cycle of modelling are over. The UTM deletes the mark 11 and a new cycle of modelling begins:

\[ 0111R0101 (q_{130}00Rq_{10}, q_{101}0Rq_{13}, q_{131}1Rq_{24}, q_{241}0Lq_{3}, q_{310}Lq_{2}, q_{201}Rq_{1}, q_{100}Rq_{5}, q_{501}Rq_{1}). \]

5. The UTM with 10 states and 3 symbols

The symbols of the machine in UTM(10,3) (see [14]) are 0 (blank symbol), 1 and \( b \); the states are \( q_i \) (\( i = 1, \ldots, 10 \)).

\[ N_1 = 2, \quad N_{k+1} = N_k + m_k + t_k \quad (k \in \{1, \ldots, n\}), \]
where if \( m_k \) is even, then \( t_k = 2 \) else \( t_k = 1 \). Obviously, all \( N_j \) (\( j \in \{1, \ldots, n + 1\} \)) are even.

The code of the production \( \alpha_i = a_{i1}a_{i2} \ldots a_{im_i} \) (\( i \in \{1, \ldots, n\} \)) of the tag-system, where \( m_i \) is odd, is

\[
P_i = b0b00^{N_{m_i}}00b00^{N_{m_i}-1}00b \ldots 00b00^{N_i}0.
\]

If \( m_i \) is even, then

\[
P_i = b0b00^{N_{m_i}}00b00^{N_{m_i}-1}00b \ldots 00b00^{N_i}0.
\]

where \( A_j = 0^{N_j} \), \( j \in \{1, \ldots, n + 1\} \) and the pair \( b0 \) is a mark.

\[
P_0 = b0b0, \quad P_{n+1} = 10.
\]

The code \( S \) of the initial word \( \beta = a_r a_s a_t \ldots a_w \), to be transformed by the tag-system, is

\[
S = 1^{N_r}bb1^{N_s}bb1^{N_t} \ldots bb1^{N_w},
\]

where \( \tilde{A}_j = 1^{N_j} \), \( j \in \{1, \ldots, n + 1\} \) and the pair \( bb \) is a mark.

The program of the machine in UTM(10,3):

\[
\begin{align*}
qu_101Rq_1 & \quad q_{200Lq_3} & \quad q_{300Lq_2} & \quad q_{401Rq_1} & \quad q_{50bLq_3} \\
qu_{10Lq_2} & \quad q_{210Lq_2} & \quad q_{31bLq_6} & \quad q_{411Rq_5} & \quad q_{511Rq_5} \\
qu_{1bbRq_4} & \quad q_{2bbLq_2} & \quad q_{3bbRq_1} & \quad q_{4b1Lq_4} & \quad q_{5bbRq_5} \\
qu_{601Lq_7} & \quad q_{200Rq_8} & \quad q_{301Lq_6} & \quad q_{401Lq_10} & \quad q_{500bRq_5} \\
qu_{611Lq_6} & \quad q_{71-} & \quad q_{811Rq_8} & \quad q_{910Rq_10} & \quad q_{1010Rq_10} \\
qu_{6bbLq_6} & \quad q_{7bbLq_9} & \quad q_{8bbRq_8} & \quad q_{9b0Lq_4} & \quad q_{10bbRq_9}
\end{align*}
\]

(i) On the first stage of modelling:

\[
\begin{align*}
b1Lb0 & \quad (q_{210Lq_2}, q_{2bbLq_2}), \\
1L0 & \quad (q_{210Lq_2}), \\
00L00 & \quad (q_{200Lq_3}, q_{300Lq_2}), \\
Lb0(b1)L & \quad (q_{200Lq_3}, q_{3bbRq_1}, q_{101Rq_1}), \\
b0Rb1 & \quad (q_{1bbRq_4}, q_{401Rq_1}), \\
0R1 & \quad (q_{101Rq_1}), \\
R1(0)L & \quad (q_{101Lq_2}).
\end{align*}
\]

If the head of the UTM moves to the right and meets the mark \( bb \), then the first stage of modelling is finished, and the mark \( bb \) is changed to the pair \( 11 \):

\[
Rbb(11)L. (q_{1bbRq_4}, q_{4b1Lq_4}),
\]

the UTM writes the mark \( bb \) in \( Q_R \) and the second stage of modelling begins.
(ii) On the second stage of modelling:

\begin{align*}
bbLbb & \quad (q_6bbLq_6), \\
blb1 & \quad (q_611Lq_6, q_6bbLq_6), \\
1L & \quad (q_611Lq_6).
\end{align*}

\[ L00(01)R \quad (q_601Lq_7, q_700Rq_8, q_811Rq_8) \]

and the UTM writes the symbol 1 in \( Q_R \).

In this case

\begin{align*}
\frac{bRb}{lRl} & \quad (q_8bbRq_8), \\
R0(1)L & \quad (q_801Lq_6).
\end{align*}

\[ L00b0(01b1)R \quad (q_601Lq_7, q_7bbLq_9, q_901Lq_{10}, q_{10}00Rq_5) \]

and the UTM writes the mark \( bb \) in \( Q_R \). In this case

\begin{align*}
\frac{bRb}{lRl} & \quad (q_5bbRq_5), \\
R10(bb)L & \quad (q_511Rq_5, q_50bLq_3, q_31bLq_6).
\end{align*}

If the head of the UTM moves to the left and meets the pair \( 10 = P_{n+1} \), then the UTM halts \( (q_601Lq_7, q_71-) \).

If the head of the UTM moves to the left and meets the group \( b0b0 \), then the second stage of modelling is over:

\[ Lb0b0(b0b0)R \quad (q_601Lq_7, q_7bbLq_9, q_901Lq_{10}, q_{10}bbRq_9, q_910Rq_{10}). \]

(iii) On the third stage of modelling:

\begin{align*}
b1Rb0 & \quad (q_{10}bbRq_9, q_910Rq_{10}), \\
1R0 & \quad (q_{10}10Rq_{10}).
\end{align*}

When the head of the UTM moves to the right and meets the mark \( bb \), both the third stage and the whole cycle of modelling are over. The UTM deletes the mark \( bb \) and a new cycle of modelling begins:

\[ Rbb(10)L \quad (q_{10}bbRq_9, q_9b0Lq_4, q_4bb1Lq_4) \]

then in some steps the head of the UTM will be located on the left side of \( S \).

6. The UTM with 7 states and 4 symbols

The symbols of the machine in UTM(7,4) (see [13]) are 0 (blank symbol), 1, \( b \) and \( c \); the states are \( q_i \ (i \in \{1,\ldots,7\}) \).

\[ N_1 = 1, \quad N_{k+1} = N_k + m_k + 1 \quad (k \in \{1,\ldots,n\}). \]

The code of the production \( \omega_i = a_{i1}a_{i2}\ldots a_{im_i} \) \( (i \in \{1,\ldots,n\}) \) of the tag system is

\[ P_i = bb00^{N_{m_i}}b00^{N_{m_i}-1}\ldots b00^{N_2}b00^{N_1}, \]
where $A_j = 0^N_j$, $j \in \{1, \ldots, n+1\}$, and the symbol $b$ is a mark.

$$P_0 = b0, \quad P_{n+1} = 10.$$  

The code $S$ of the initial word $\beta = a_1a_2a_3 \ldots a_w$, to be transformed by the tag-system, is

$$S = 1^{N_c}c1^{N_c}c1^{N_c} \ldots c1^{N_c},$$

where $\tilde{A}_j = 1^{N_j}$, $j \in \{1, \ldots, n+1\}$, and the symbol $c$ is a mark.

The program of the machine in UTM(7,4):

\[
\begin{align*}
q_110Lq_1 & \quad q_201Rq_2 \quad q_301Lq_4 \quad q_401Lq_7 \\
q_110Lq_1 & \quad q_210Lq_1 \quad q_311Rq_3 \quad q_411Lq_4 \\
q_1bcRq_2 & \quad q_2bcRq_2 \quad q_3bcRq_3 \quad q_4bcLq_4 \\
q_1cbLq_1 & \quad q_2c1Rq_5 \quad q_3cbRq_3 \quad q_4cbLq_4 \\
q_50cLq_4 & \quad q_600Rq_5 \quad q_700Rq_3 \\
q_511Rq_5 & \quad q_610Rq_6 \quad q_71- \\
q_5bcRq_5 & \quad q_6bbRq_6 \quad q_7bbLq_6 \\
q_5cbRq_5 & \quad q_5c0Rq_1 \quad q_7c- \\
\end{align*}
\]

(i) On the first stage of modelling:

$1Lb0$ \quad ($q_110Lq_1, \ q_1cbLq_1$),

$1L0$ \quad ($q_110Lq_1$),

$0L0$ \quad ($q_100Lq_1$),

$b0Rc1$ \quad ($q_2bcRq_2, \ q_201Rq_2$),

$Lb0(c1)L$ \quad ($q_100Lq_1, \ q_1bcRq_2, \ q_201Rq_2$),

$0R1$ \quad ($q_201Rq_2$),

$R1(0)L$ \quad ($q_210Lq_1$).

If the head of the UTM moves to the right and meets the mark $c$, then the first stage of modelling is finished, and the mark $c$ is changed to the symbol 1:

$cR1$ \quad ($q_2c1Rq_5$),

and the UTM writes the mark $c$ in $Q_R$ and the second stage of modelling begins.

(ii) On the second stage of modelling:

$1Lb$ \quad ($q_4cbLq_4$),

$1LC$ \quad ($q_4bcLq_4$),

$1L1$ \quad ($q_411Lq_4$).
$L00(01)R$ ($q_401Lq_7$, $q_700Rq_3$, $q_311Rq_3$) and the UTM writes the symbol 1 in $Q_R$. In this case:

- $cRb$ ($q_3cbRq_3$),
- $bRc$ ($q_3bcRq_3$),
- $1R1$ ($q_311Rq_3$),
- $R0(1)L$ ($q_301Lq_4$).

$L0b0(0c1)R$ ($q_401Lq_7$, $q_7bbLq_6$, $q_600Rq_5$, $q_5bcRq_5$, $q_511Rq_5$) and the UTM writes the mark c in $Q_R$. In this case:

- $cRb$ ($q_5cbRq_3$),
- $bRc$ ($q_5bcq_5$),
- $1R1$ ($q_511Rq_5$),
- $R0(c)L$ ($q_50cLq_4$).

If the head of the UTM moves to the left and meets the pair $10 = P_{n+1}$, then the UTM halts ($q_401Lq_7$, $q_71-1$).

If the head of the UTM moves to the left and meets the group $bb0$, then the second stage of modelling is over:

$Lbb0(bb0)R$ ($q_401Lq_7$, $q_7bbLq_6$, $q_6bbRq_6$, $q_610Rq_6$).

(iii) On the third stage of modelling:

- $b1Rb0$ ($q_6bbRq_6$, $q_610Rq_6$),
- $1R0$ ($q_610Rq_6$).

When the head of the UTM moves to the right and meets the mark c, both the third stage and the whole cycle of modelling are over. The UTM deletes the mark c and a new cycle of modelling begins:

- $cR0$ ($q_6c0Rq_1$).

## 7. The UTM with 5 states and 5 symbols

The machine in UTM(5,5) (see [13]) simulates the following class of tag-systems:

\[
(T_1) \begin{cases} 
  a_i \rightarrow b\alpha_i a, & i \in \{1, \ldots, n\} \\
  a \rightarrow \Lambda \\
  b \rightarrow \Lambda \\
  a_{n+1} \rightarrow STOP
\end{cases}
\]

where $\alpha_i = a_{i1}a_{i2}\ldots a_{im}$ is a finite word in the alphabet $A = \{a_k\}$, $k \in \{1, \ldots, n+1\}$ ($\alpha_i$ is not empty) and $\Lambda$ is the empty word.

We show the universality of the tag-systems of type $T_1$ in Lemma 7.1.
**Lemma 7.1.** For every tag-system $T$ of type $T$ (see Definition 5), there is a tag-system $T'$ of type $T_1$ which models $T$.

**Proof.** We change a fixed tag-system $T$ by adding two new letters $a$ and $b$ with

$$
\begin{align*}
    a_i & \rightarrow bx_ia, \quad i \in \{1, \ldots, n\} \\
    a & \rightarrow \Lambda \\
    b & \rightarrow \Lambda \\
    a_{n+1} & \rightarrow \text{STOP}
\end{align*}
$$

We show that if $\beta \xrightarrow{T} \gamma$, then $\beta a \xrightarrow{T'} \gamma'a$ (the first letter in the word $\gamma'$ is different from $a$ and $b$) and $\gamma$ results from $\gamma'$ by deleting any occurrences of the symbols $a$ and $b$.

**Induction basis.** Let $a_i \beta \xrightarrow{T} \beta a_i$. Then $a_i a_j \beta a \xrightarrow{T'} \beta abx_ia$. If $\beta$ is empty, then $a_i a_j \xrightarrow{T'} abx_ia \xrightarrow{T'} a_i a$. Because $a \neq \Lambda$, the basis is proved.

**Induction hypothesis.** Let $\beta \xrightarrow{T} \gamma_i \xrightarrow{T} \gamma$ and $\beta a \xrightarrow{T'} \gamma'a$, where the first letter in the word $\gamma'$ is different from $a$ and $b$, and $\gamma_i$ results from $\gamma'$ by deleting any occurrences of $a$ and $b$.

We note that in the process of transforming of word $\beta a$ by the tag-system in the sequence $\beta a \xrightarrow{T'} \beta_1 a \xrightarrow{T'} \ldots \xrightarrow{T'} \beta_j a \xrightarrow{T'} \ldots$ the words that do not begin with the symbols $a$ and $b$ have the form: $\beta_j = \beta_j a \beta_j ab \ldots \beta_{j_{r+1}} a \beta_{j_{k_{r}}} \ldots (\beta_{j_{r}}, r \in \{1, \ldots, k_j\}$, the symbols $a$ and $b$ do not occur, and that the words $\beta_{j_{1}}$ and $\beta_{j_{k_{1}}}$ are not empty).

Consider two cases:

(i) $\gamma'_i = a_i a_\delta i$, $i, j \in \{1, 2, \ldots, n+1\}$, $i \neq n+1$. Then $\gamma_i = a_i a_\delta i$, where $\delta_i$ results from $\delta'_i$ by deleting any occurrences of the symbols $a$ and $b$, and $a_i a_\delta i \xrightarrow{T'} \delta_i x_i \quad (\gamma = \delta_i x_i)$, though $a_i a_\delta i \xrightarrow{T'} \delta'_i abx_i a$.

If $\delta_i$ is empty, then we take into account that $\delta'_i$ is also empty. Then $\delta'_i abx_i a = abx_i a \xrightarrow{T'} x_i a$. Let $\gamma' = x_i$. This is a desirable $\gamma'$.

If $\delta_i$ is not empty, then we take into account that either the first letter in the word $\delta'_i$ is different from $a$ and $b$ (then $\gamma' = \delta'_i abx_i$) or $\delta'_i = a_\delta i$ (where $\delta_i$ results from $\gamma'$ by deleting any occurrences of $a$ and $b$). In the latter case $a_\delta i ab x_i a \xrightarrow{T'} \delta'_i abx_i a$. Let $\gamma' = \delta'_i abx_i$. This is a desirable $\gamma'$.

(ii) $\gamma'_i = a_i a_\delta i$, $i \in \{1, 2, \ldots, n\}$. Then $\delta'_i = b \delta'_i$ and $\gamma_i = a_i a_\delta i$, where $\delta_i$ results from $\delta'_i$ by deleting any occurrences of $a$ and $b$.

Let be $\delta_i = a_\delta i$, $j \in \{1, 2, \ldots, n+1\}$. It means $a_i a_\delta i \xrightarrow{T} \delta_i x_i \quad (\gamma = \delta_i x_i)$. Taking into account that $\delta'_i = a_i a_\delta i$ and $\delta_{i+1}$ results from $\delta'_{i+1}$ by deleting any occurrences of $a$ and $b$, we have

$$
a_i a_\delta i \xrightarrow{T'} b i \delta'_{i+1} abx_i a \xrightarrow{T'} \delta'_i abx_i a.
$$
If $\delta_{i1}$ is empty, then $\delta'_{i1}$ is empty as well. Then $\delta'_{i1}ab\alpha_i a \rightarrow \alpha_ia$ and $\gamma' = \alpha_i$. If $\delta_{i1}$ is not empty, then either $\delta'_{i1}$ begins with a letter other than $a$ and $b$ (then $\gamma' = \delta'_{i1}ab\alpha_i$), or $\delta'_{i1} = ab\delta''_{i1}$, where the word $\delta''_{i1}$ begins with a letter other than $a$ and $b$, and $\delta_{i1}$ results from $\delta''_{i1}$ by deleting any occurrences of $a$ and $b$. In the latter case, $ab\delta''_{i1}ab\alpha_i a \rightarrow \delta'_{i1}ab\alpha_i a$. Let $\gamma' = \delta''_{i1}ab\alpha_i$. This is a desirable $\gamma'$. □

The symbols of the machine in UTM(5,5) are 0, 1, $b$ (blank symbol), $c$ and $d$; and the states are $q_i$ ($i = 1, \ldots, 5$).

$N_1 = 3$, $N_{k+1} = N_k + m_k + 4$ ($k \in \{1, \ldots, n\}$).

$N_a = N_{n+1} + 2$ and an arbitrary number $N > N_a$.

$N_b = 1$.

The code of the production $\alpha_i = b\alpha_{i1}a_{i2}\ldots a_{im}a$ ($i \in \{1, \ldots, n\}$) of the tag-system is

$$P_i = bbl^{N_0}1bl^{N_{m_i}}1bl^{N_{m_{i-1}}}b\ldots 1bl^{N_i}1bl^{N_b}1b,$$

where $A_j = 1^{N_j}$, $j \in \{1, \ldots, n+1\}$, $A = 1^{N_a}$, $B = 1^{N_b}$ and the symbol $b$ is a mark.

$P_0 = bbb$, $P_{n+1} = 1blb$.

The code $S$ of the initial word $\beta = a_r a_s a_t a\ldots a_w$, to be transformed by the tag-system, is

$$S = 1^{N_s}c1^{N_s}c1^{N_t}\ldots c1^{N_w}c1^{N_r},$$

where $A_j = 1^{N_j}$, $j \in \{1, \ldots, n+1\}$, $A = 1^{N_a}$, $B = 1^{N_b}$ and the symbol $c$ is a mark.

The program of the machine in UTM(5,5):

$$q_101Rq_1 \quad q_200Rq_2 \quad q_30cLq_4 \quad q_401Lq_4 \quad q_50-$$
$$q_110Lq_1 \quad q_210Rq_2 \quad q_310Rq_3 \quad q_410Rq_2 \quad q_511Rq_5$$
$$q_1bdRq_1 \quad q_2b0Lq_4 \quad q_3bbRq_5 \quad q_4bdLq_3 \quad q_5b-$$
$$q_1c0Rq_2 \quad q_2ccRq_2 \quad q_3ccRq_3 \quad q_4ccLq_4 \quad q_5c1Rq_3$$
$$q_1dbLq_1 \quad q_2ddRq_2 \quad q_3ddRq_3 \quad q_4ddLq_4 \quad q_5dbRq_5$$

(i) On the first stage of modelling:

$$dLb \quad (q_1dbLq_1),$$
$$1L0 \quad (q_110Lq_1),$$
$$Lb(d)R \quad (q_1bdRq_1),$$
$$0R1 \quad (q_101Rq_1),$$
$$bRd \quad (q_1bdRq_1),$$
$$R1(0)L \quad (q_110Lq_1).$$
If the head of the UTM moves to the right and meets the mark $c$, the first stage of modelling is finished. Then the mark $c$ is changed to the symbol 0:

$$cR0 \ (q_1c0Rq_2),$$

and the second stage of modelling begins.

(ii) On the second stage of modelling:

$$dLb \ (q_4ddLq_4),$$
$$cLc \ (q_4ccLq_4),$$
$$0L1 \ (q_401Lq_4).$$

$L1(0)R \ (q_410Rq_2)$ and the UTM writes the symbol 0 in $Q_R$. In this case

$$dRd \ (q_2ddRq_2),$$
$$cRc \ (q_2ccRq_2),$$
$$1R0 \ (q_210Rq_2),$$
$$(0R0 \ (q_200Rq_2) \text{ if the UTM wrote the symbol 0 before that),}$$

then $Rb(0)L \ (q_3b0Lq_4)$.

As after the first stage of modelling $P'_k \ (k = 0, \ldots, r - 1)$ differs from $P_k$ by the fact that the marks $b$ are changed to the marks $d$, the UTM will write a number of the symbols 0 in $Q_R$ as many as the number of the symbols 1 between the codes $P_r$ and $S$. As a result there will be a new code $A$ of the symbol $a$.

$L1b(0d)R \ (q_4bdLq_3, q_310Rq_3, q_3ddRq_3)$ and the UTM writes the mark $c$ in $Q_R$. In this case

$$dRb \ (q_3ddRq_3),$$
$$cRc \ (q_3ccRq_3),$$
$$1R0 \ (q_310Rq_3),$$
$$RO(c)L \ (q_30cLq_4).$$

If the head of the UTM moves to the left and meets the group $1b1h = P_{n+1}$, then the UTM will try to write two marks $c$ in $Q_R$ and halts ($q_3bbRq_5, q_5b-$).

If the head of the UTM moves to the left and meets the pair $bb$, the second stage of modelling is over:

$Lbb(bb)R \ (q_4bdLq_3, q_3bbRq_5, q_5dbRq_5)$.

If, at the beginning of a new cycle of modelling $A_r = A$, then on the first stage of modelling the head of the UTM goes to $Q_L$ without a failure and on the second stage of modelling the UTM will write the number of 0 in $Q_R$ as many times as the number of 1s that are in $P$. As a result there will be a new code $A$ of the symbol $a$. Then, on the second stage the UTM meets the pair $bb$ in $Q_L$ without a failure and the third stage of modelling begins.

If at the beginning of a new cycle of modelling $A_r = B$, then on the first stage of modelling the UTM meets the pair $bb$ in $P_0$ and turns immediately to the third stage of modelling.
(iii) On the third stage of modelling:

\[ dRb \quad (q_5dbRq_s), \]
\[ 1R1 \quad (q_511Rq_s). \]

When the head of the UTM moves to the right and meets the mark \( c \), both the third stage and the whole cycle of modelling are over. The UTM deletes the mark \( c \) and a new cycle of modelling begins:

\[ cR1 \quad (q_5c1Rq_1). \]

8. The UTM with 4 states and 6 symbols

The machine in UTM(4,6) (see [13]) simulates the following class of tag-systems:

\[ (T_2) \quad \begin{cases} 
  a_i \to a_i, & i \in \{1, \ldots, n\} \\
  a_n \to a_na_n \\
  a_{n+1} \to \text{STOP} 
\end{cases} \]

where \( a_i = a_ia_n\beta_i \) and \( \beta_i \) are not empty.

We show the universality of tag-systems of type \( T_2 \) in Lemma 8.1.

**Lemma 8.1.** For every tag-system \( T \) of type \( T \), there is the tag-system \( T' \) of type \( T_2 \) which models \( T \).

**Proof.** Can be provided in the same manner as that of Lemma 7.1. \( \Box \)

The symbols of the machine in UTM(4,6) are \( 0 \) (blank symbol), \( 1 \), \( b \), \( \tilde{b} \), \( \tilde{b} \) and \( c \); the states are \( q_i \) (\( i = 1, \ldots, 4 \)).

\[ N_1 = 1, \quad N_{k+1} = N_k + 2m_k \quad (k \in \{1, \ldots, n\}). \]

We note that \( m_n = 2 \). The code of the production \( a_i = a_na_na_{i3} \ldots a_{im_i} \) (\( i \in \{1, \ldots, n\} \)) of the tag-system is

\[ P_i = b1b1^{N_{m_i}}bb1^{N_{m_i}-1} \ldots bb1^{N_{j}}bb1^{N_e}bb1^{N_e-N_i}. \]

In particular,

\[ P_n = b1b1^{N_e}bb, \quad P_0 = b, \quad P_{n+1} = \tilde{b}bb, \]

where \( A_j = 1^{N_j}, j \in \{1, \ldots, n+1\} \) and the symbol \( b \) is a mark.

The code \( S \) of the initial word \( \beta = a_1a_2a_3 \ldots a_w \), to be transformed by the tag-system, is

\[ S = 1^{N_e}c1^{N_e}c1^{N_e} \ldots c1^{N_e} \]

and the symbol \( c \) is a mark.
The program of the machine in UTM(4,6):

\[ q_1 \overrightarrow{b} L q_1 \quad q_2 \ 10 \ R q_2 \quad q_3 \ 11 \ R q_3 \quad q_4 \ 10 \ R q_4 \\
q_1 \ b \ b \ R q_1 \quad q_2 \ b \ b \ L q_3 \quad q_3 \ b \ b \ R q_4 \quad q_4 \ b c \ L q_2 \\
q_1 \ b b \ L q_1 \quad q_2 \ b b \ R q_2 \quad q_3 \ b b \ R q_3 \quad q_4 \ b b \ R q_4 \\
q_1 \ 0 b \ R q_1 \quad q_2 \ 01 \ L q_2 \quad q_3 \ 0 c \ R q_1 \quad q_4 \ 0 c \ L q_2 \\
q_1 \ c 0 \ R q_4 \quad q_2 \ c b \ R q_2 \quad q_3 \ c 1 \ R q_1 \quad q_4 \ c b \ R q_4 \\
\]

(i) On the first stage of modelling:

- \[ 1 \ L \ b \ (q_1 \overrightarrow{b} L q_1), \]
- \[ 0 \ L \ b \ (q_1 \overrightarrow{b} L q_1), \]
- \[ b \ L \ b \ (q_1 \overrightarrow{b} L q_1), \]
- \[ L \ b \ (b) R \ (q_1 \overrightarrow{b} R q_1), \]
- \[ \overrightarrow{b} \ R \ 0 \ (q_1 \overrightarrow{b} R q_1), \]
- \[ b \ R \ b \ (q_1 \overrightarrow{b} R q_1), \]
- \[ R \ 1 \ (b) L \ (q_1 \overrightarrow{b} L q_1). \]

If the head of the UTM moves to the right and meets the mark \( c \), the first stage of modelling is finished. At that time the tape of the UTM is

\[ QL_{P_n} P_{n+1} P_1 P_2 P_3 \ldots P_{r-1} P_r P_r' \ldots P'_1 P'_1 R' A'_n A'_1 \ldots A_w Q_R \]

and the head of the UTM locates the symbol \( c \) between the codes of \( A'_r \) and \( A_s \) (\( A'_j = 0^N, \ j \in \{1, \ldots, n+1\} \)). \( P'_k (k \in \{0, \ldots, r-1\}) \) differs from \( P_k \) by the fact that the marks \( b \) are changed to the marks \( \overrightarrow{b} \) and \( A_j \) to \( A'_j \) (\( j \in \{1, \ldots, n+1\} \)):

\[ P'_k = \overrightarrow{b} b \overline{b} b \overline{b}^{N_{n+1}} \overrightarrow{b} \overline{b} b \overline{b}^{N_{n+1}} - N_1. \]

After that the UTM deletes the mark \( c (q_1 c 0 R q_4) \) and the second stage of modelling begins (first of all, the UTM writes the mark \( c \) in the part \( Q_R \)).

(ii) On the second stage of modelling:

- \[ 0 \ L 1 \ (q_2 \ 01 \ L q_2), \]
- \[ \overrightarrow{b} \ L \overrightarrow{b} \ (q_2 \overrightarrow{b} L q_2), \]
- \[ 0 \ b \ L 1 \ c \ (q_2 \overrightarrow{b} L q_3, \ q_3 \ 0 c \ R q_1, \ q_1 \overrightarrow{b} L q_1, \ q_1 \ c 0 \ R q_4, \ q_4 \ b c \ L q_2, \ q_2 \ 01 \ L q_2). \]

\[ L \overrightarrow{b} \overrightarrow{b} R \ (q_2 \overrightarrow{b} \overrightarrow{b} R q_2) \] and the UTM writes symbol 1 in \( Q_R \).

As after the first stage of modelling there are exactly \( N_r \) marks \( \overrightarrow{b} \) between the codes \( P_r \) and \( S \), the UTM will write exactly \( N_r \) symbols 1 in \( Q_r \). After that, when the head is located on the code \( P_r \), the UTM will write \( N_n - N_r \) more symbols 1 in \( Q_r \). As a result there will be the code \( A_n \).
$L1(0)R \ (q_2 10 Rq_2)$ and the UTM writes the symbol 1 in $Q_R$. In this case

\[
\begin{align*}
1 \ R \ 0 & \ (q_2 \ 10 \ Rq_2), \\
\bar{b} \ R \ \bar{b} & \ (q_2 \ \bar{b}\bar{b} \ Rq_2), \\
c \ R \ b & \ (q_2 \ cb \ Rq_2), \\
R \ 0 \ (1)L & \ (q_2 \ 01 \ Lq_2).
\end{align*}
\]

$Lbb(\bar{b}b)R \ (q_2 \ b\bar{L}q_3, \ q_3 \ b\bar{b}Rq_4, \ q_4\bar{b}\bar{b}Rq_4)$ and the UTM writes the mark $c$ in $Q_R$. In this case

\[
\begin{align*}
1 \ R \ 0 & \ (q_4 \ 10 \ Rq_4), \\
\bar{b} \ R \ \bar{b} & \ (q_4 \ \bar{b}\bar{b} \ Rq_4), \\
c \ R \ b & \ (q_4 \ cb \ Rq_4), \\
R \ 0 \ (c)L & \ (q_4 \ 0c \ Lq_2).
\end{align*}
\]

If the head of the UTM moves to the left and meets the pair $\bar{b}b$, then the UTM halts ($q_2 \ b\bar{L}q_3, \ q_3 \ b\bar{L}q_3$).

If the head of the UTM moves to the left and meets the pair $1b$, the second stage of modelling is over. At that time the tape of the UTM is

\[
Q_LP_{n+1}P_n \ldots P_{r+1}P_r \ldots P_1P_0R''A_1\ldots A_nA_nA_3A_4\ldots A_mQ_R
\]

and the head is located on the second left symbol of the code $P'' \ (k \in \{0, \ldots, r\})$ differs from $P_k$ by the fact that the marks $b$ are changed to the marks $\bar{b}$.

Then $L1b(1b)R \ (q_2 \ b\bar{L}q_3, \ q_3 \ 11 \ Rq_3, \ q_3 \ b\bar{b}Rq_3)$ and the UTM goes to the third stage of modelling.

(iii) On the third stage of modelling the UTM restores the tape in $P$:

\[
\begin{align*}
\bar{b} \ R \ b & \ (q_3 \ b\bar{b} \ Rq_3), \\
1 \ R \ 1 & \ (q_3 \ 11 \ Rq_3).
\end{align*}
\]

When the head of the UTM moves to the right and meets the mark $c$, the third stage and the whole cycle of modelling are over. The UTM deletes the mark $c$ and a new cycle of modelling ($q_3c1Rq_1$) begins.

9. The UTM with 3 states and 10 symbols

The symbols of the machine in $UTM(3,10)$ (see [15]) are 0 (blank symbol), 1, $\tilde{1}, \tilde{1}, b, \bar{b}, c, \bar{c}, \tilde{c}$; and the states are $q_1, q_2, q_3$.

\[
N_1 = 1, \quad N_{k+1} = N_k + 2m_k + 2 \ (k \in \{1, \ldots, n\}).
\]
The code of the production $\alpha_i = a_{i1}a_{i2} \ldots a_{in_i}$, $(i \in \{1, \ldots, n\})$ of the tag-system is

$$P_i = b1bb1^{N_{i}}b1^{N_{i-1}} \ldots b1^{N_2}b1^{N_1},$$

where $A_j = 1^{N_j}$, $j \in \{1, \ldots, n + 1\}$ and the symbol $b$ is a mark.

$$P_0 = b, \quad P_{n+1} = cb.$$ 

The code $S$ of the initial word $\beta = a_1a_2a_3 \ldots a_m$, to be transformed by the tag-system, is

$$S = 1^{N_c}c1^{N_c}c1^{N_c} \ldots c1^{N_c}c,$$

where the symbol $c$ is a mark.

The program of the machine in UTM(3,10):

\[
\begin{align*}
q_1 & 0c \quad Lq_3 & q_2 & 0\bar{c} \quad Lq_2 & q_3 & 0 \quad - \\
q_1 & \text{b}\bar{b} \quad Rq_1 & q_2 & \text{b}\bar{b} \quad Lq_3 & q_3 & \text{b}\bar{b} \quad Rq_1 \\
q_1 & \text{b}\bar{b} \quad Lq_1 & q_2 & \text{b}\bar{b} \quad Lq_2 & q_3 & \text{b}\bar{b} \quad Lq_2 \\
q_1 & \text{c}\bar{b} \quad Rq_1 & q_2 & \text{c}\bar{b} \quad Rq_2 & q_3 & \text{c}\bar{b} \quad Rq_2 \\
q_1 & \text{c}\bar{b} \quad Lq_1 & q_2 & \text{c}\bar{b} \quad Lq_2 & q_3 & \text{c}\bar{b} \quad Lq_2 \\
q_1 & \text{c}\bar{c} \quad Rq_1 & q_2 & \text{c}\bar{c} \quad Rq_2 & q_3 & \text{c}\bar{c} \quad Rq_2 \\
q_1 & \text{c}\bar{c} \quad c \quad Rq_1 & q_2 & \text{c}\bar{c} \quad c \quad Rq_3 & q_3 & \text{c}\bar{c} \quad Rq_3 \\
q_1 & \text{c}\bar{c} \quad - \quad Rq_1 & q_2 & \text{c}\bar{c} \quad c \quad Rq_3 & q_3 & \text{c}\bar{c} \quad Lq_3 \\
\end{align*}
\]

(i) On the first stage of modelling:

\[
\begin{align*}
1 & L \bar{c} & (q_1 \text{c}\bar{c} \quad Lq_1), \\
\bar{c} & L \bar{c} & (q_1 \text{c}\bar{c} \quad Lq_1), \\
\text{b} & b & (q_1 \text{b}\bar{b} \quad Lq_1), \\
L & b \quad (\text{b}\bar{b} \quad Rq_1), \\
\bar{c} & R \bar{c} & (q_1 \text{c}\bar{c} \quad Rq_1), \\
b & R \text{b} & (q_1 \text{b}\bar{b} \quad Rq_1), \\
R & 1 \quad (\bar{c}\bar{c}) & (q_1 \text{c}\bar{c} \quad Lq_1).
\end{align*}
\]

If the head of the UTM moves to the right and meets the mark $c$, the first stage of modelling is over. The UTM deletes this mark and the second stage of modelling begins $(q_1 \text{c} \quad \bar{c}Lq_2)$.

(ii) On the second stage of modelling the UTM writes the marks $c$ and the symbols $\bar{c}$ in $QR$; moreover the UTM writes the mark $c$ only after the symbol $\bar{c}$. 


If the UTM writes the symbol $\bar{1}$ in $Q_R$ either after writing the mark $c$ or after the first stage of modelling, then

\begin{align*}
\bar{b} & R \bar{b} \quad (q_2 \bar{b}\bar{b} Rq_2), \\
\bar{1} & R \bar{1} \quad (q_2 \bar{1}\bar{1} Rq_2), \\
1 & R \bar{1} \quad (q_2 \bar{1}1 Rq_2), \\
\bar{c} & R \bar{c} \quad (q_2 \bar{c}\bar{c} Rq_2), \\
R & 0 (\bar{1})L \quad (q_2 0\bar{1} Lq_2).
\end{align*}

If the UTM writes the symbol $\bar{1}$ in $Q_R$ after the same symbol $\bar{1}$, then

\begin{align*}
\bar{b} & R \bar{b} \quad (q_2 \bar{b}\bar{b} Rq_2), \\
\bar{1} & R \bar{1} \quad (q_2 \bar{1}\bar{1} Rq_2), \\
\bar{c} & R \bar{c} \quad (q_2 \bar{c}\bar{c} Rq_2), \\
R & 0 (\bar{1})L \quad (q_2 0\bar{1} Lq_2).
\end{align*}

The UTM goes to the left after writing the symbol $\bar{1}$ in $Q_R$:

\begin{align*}
\bar{1} & L \bar{1} \quad (q_2 \bar{1}\bar{1} Lq_2), \\
\bar{c} & L \bar{c} \quad (q_2 \bar{c}\bar{c} Lq_2), \\
\bar{b} & L \bar{b} \quad (q_2 \bar{b}\bar{b} Lq_2).
\end{align*}

Then, if the head of the UTM meets the symbol $1$ in $P_r$, the head will change the direction of its motion, the UTM changes the symbol $1$ to the symbol $\bar{1}$ and writes the symbol $\bar{1}$ in $Q_R$:

\[ L 1(\bar{1})R \quad (q_2 1\bar{1} Rq_2). \]

If the head of the UTM meets the marks $bb$ in $P_r$, then the UTM writes the mark $c$ in $Q_R$:

\[ L bb (\bar{b}\bar{b})R \quad (q_2 b \bar{b}Lq_3, q_3 b \bar{b}Rq_1, q_1 b \bar{b}Rq_1), \]

then

\begin{align*}
\bar{b} & R \bar{b} \quad (q_1 \bar{b}\bar{b} Rq_1), \\
\bar{1} & R \bar{1} \quad (q_1 \bar{1}\bar{1} Rq_1), \\
\bar{c} & R \bar{c} \quad (q_1 \bar{c}\bar{c} Rq_1), \\
R & 0 (c)L \quad (q_1 0c Lq_3).
\end{align*}

When the head of the UTM moves to the left after writing the symbol $c$ in $Q_R$, then

\begin{align*}
\bar{1} & L 1 \quad (q_3 \bar{1}1 Lq_3), \\
\bar{c} & L c \quad (q_3 \bar{c}c Lq_3).
\end{align*}
and the UTM restores the part $S$ of the tape. Then the UTM meets the mark $\overline{b}$ in $P$:

\[
\begin{align*}
&\overline{b} L \overline{b} \quad (q_3 \overline{b} L q_2, \quad q_2 \overline{b} L q_2), \\
&\overline{1} L \overline{1} \quad (q_2 \overline{1} L q_2).
\end{align*}
\]

The UTM halts when it meets the pair $\overline{c} b$ ($q_2 b L q_3$, $q_3 \overline{c} c$).

If the UTM meets the pair $1b$, then the second stage of modelling is over. Then

\[
L 1b(1b) \quad (q_2 \overline{b} L q_3, \quad q_3 11 R q_3, \quad q_3 \overline{b} R q_3)
\]

and the third stage of modelling begins.

(iii) On the third stage of modelling the UTM restores the tape in $P$ (the tape is restored in $S$ after writing the mark $c$ in $Q_R$):

\[
\begin{align*}
&\overline{1} R 1 \quad (q_3 \overline{1} L q_3), \\
&1 R 1 \quad (q_3 11 R q_3), \\
&\overline{b} R b \quad (q_3 \overline{b} R q_3).
\end{align*}
\]

When the head of the UTM moves to the right and meets the mark $c$, both the third stage and the whole cycle of modelling are over. The UTM deletes the mark $c$ and a new cycle of modelling begins ($q_3c1Rq_1$).

10. The UTM with 2 states and 18 symbols

The symbols of the machine in $UTM(2,18)$ are $\overline{1}$ (blank symbol), 1, $\overline{1}$, $\overline{1}$, $1$, $b$, $\overline{b}$, $\overline{b}$, $b_1$, $\overline{b}_1$, $b_2$, $b_3$, $c$, $\overline{c}$, $\overline{c}$, $c_1$, $\overline{c}_1$, $c_2$, and the states are $q_1$ and $q_2$.

\[
N_1 = 1, \quad N_{k+1} = N_k + m_k + 1 \quad (k \in \{1,\ldots,n\}).
\]

The code of the production $a_i = a_1 a_2 \ldots a_{i_m}$ ($i \in \{1,\ldots,n\}$) of the tag-system is

\[
P_i = bb_1^{N_{i_1}}b_1^{N_{i_2}} \ldots b_1^{N_{i_{m-1}}}b_1^{N_{i_m}}1b_{i_r}^{N_{i_1}},
\]

where $A_j = 1^{N_j}$, $j \in \{1,\ldots,n+1\}$ and the symbol $b$ is a mark.

\[
P_0 = bb, \quad P_{n+1} = \overline{c}_1 \overline{c}_1.
\]

The code $S$ of the initial word $\beta = a_1 a_2 a_3 \ldots a_w$, to be transformed by the tag-system, is

\[
S = 1^{N_r}c1^{N_r}c1^{N_1} \ldots c1^{N_w}c,
\]

and the symbol $c$ is a mark.
The program of the machine in UTM(2,18):

\[ q_1 \ 1c_2 \ Lq_1 \ q_2 \ 1 \ Rq_2 \]
\[ q_1 \ \bar{1}c_2 \ Rq_1 \ q_2 \ \bar{1} \ Rq_2 \]
\[ q_1 \ \bar{1}c_2 \ Lq_1 \ q_2 \ \bar{1} \ Lq_2 \]
\[ q_1 \ \bar{1}c_2 \ Rq_1 \ q_2 \ \bar{1} \ Rq_2 \]
\[ q_1 \ bb \ Rq_1 \ q_2 \ bb \ Rq_1 \]
\[ q_1 \ bb \ Lq_1 \ q_2 \ bb \ Lq_2 \]
\[ q_1 \ \bar{b}_1 \ Rq_1 \ q_2 \ \bar{b}_1 \ Rq_2 \]
\[ q_1 \ \bar{b}_1 \ Lq_1 \ q_2 \ \bar{b}_1 \ Lq_2 \]
\[ q_1 \ b_2 b_3 \ Lq_2 \ q_2 \ b_2 b_3 \ Rq_1 \]
\[ q_1 \ b_2 b_3 \ Lq_2 \ q_2 \ b_2 b_3 \ Rq_2 \]
\[ q_1 \ c_1 \ Lq_2 \ q_2 \ c_1 \ Rq_2 \]
\[ q_1 \ c_1 \ Rq_1 \ q_2 \ c_1 \ Rq_2 \]
\[ q_1 \ c_1 \ \bar{c}_1 \ Rq_2 \ q_2 \ c_1 \ \bar{c}_1 \ Lq_1 \]
\[ q_1 \ c_2 \ Lq_1 \ q_2 \ c_2 \ Lq_2 \]

(i) On the first stage of modelling:

\[ 1 \ L \ c_2 \ (q_1 \ 1c_2 \ Lq_1), \]
\[ \bar{1} \ L \ c_2 \ (q_1 \ \bar{1}c_2 \ Lq_1), \]
\[ \bar{b} \ L \ b \ (q_1 \ \bar{b}b \ Lq_1), \]
\[ L \ b \ (\bar{b})R \ (q_1 \ \bar{b}b \ Rq_1), \]
\[ c_2 \ R \ \bar{1} \ (q_1 \ c_2 \ \bar{1} \ Rq_1), \]
\[ b \ R \ \bar{b} \ (q_1 \ \bar{b}b \ Rq_1), \]
\[ R \ 1 \ (c_2)L \ (q_1 \ 1c_2 \ Lq_1). \]

If the head of the UTM moves to the right and meets the mark \( c \), then the first stage of modelling is over. The UTM deletes this mark and the second stage of modelling begins \((q_1 \ c \ \bar{1}Lq_2)\). The tape of the UTM is

\[ Q_LP_{n+1}P_n \ldots P_{r+1}P_rP'_{r-1} \ldots P_1P'_0R'A_1 \ldots A_nQ_{R'}, \]

where in \( P'_i \) \((i \in \{0,\ldots,r-1\})\) the \( 1 \) symbols are replaced by \( \bar{1} \) and the \( b \) marks are replaced by \( \bar{b} \), \( R' \) consists of \( \bar{1} \) and \( \bar{1} \) and the head of the UTM is located on the \( R' \) in the state \( q_2 \).

(ii) On the second stage of modelling the UTM writes the marks \( c_2 \) and the symbols \( \bar{1} \) in \( Q_{R'} \); moreover, the UTM writes the mark \( c_2 \) only after the symbol \( \bar{1} \).

If the head of the UTM moves to the left in the code \( P_r \) and meets the symbol \( 1 \), then the head will change the direction of its motion, the UTM changes the symbol \( 1 \)
to the symbol $\bar{1}$ and writes the symbol $\bar{1}$ in $Q_R$:

$L \ 1(\bar{1}) \ R \ (q_2 \ \bar{1} \ \bar{1} \ Rq_2)$. 

If the UTM writes the symbol $\bar{1}$ in $Q_R$ either after writing the mark $c_2$ or after the first stage of modelling, then

$\bar{b} \ R \bar{b} \ (q_2 \ \bar{b} \ \bar{b} \ Rq_2)$,
$\bar{1} \ R \bar{1} \ (q_2 \ \bar{1} \ \bar{1} \ Rq_2)$,
$1 \ R \bar{1} \ (q_2 \ \bar{1} \ \bar{1} \ Rq_2)$,
$c \ R \bar{c} \ (q_2 \ \bar{c} \ \bar{c} \ Rq_2)$,
$R \ \bar{1} \ (\bar{1})L \ (q_2 \ \bar{1} \ \bar{1} \ Lq_2)$. 

If the UTM writes the symbol $\bar{1}$ in $Q_R$ after the same symbol $\bar{1}$, then

$\bar{b} \ R \bar{b} \ (q_2 \ \bar{b} \ \bar{b} \ Rq_2)$,
$\bar{1} \ R \bar{1} \ (q_2 \ \bar{1} \ \bar{1} \ Rq_2)$,
$\bar{c} \ R \bar{c} \ (q_2 \ \bar{c} \ \bar{c} \ Rq_2)$,
$R \ \bar{1} \ (\bar{1})L \ (q_2 \ \bar{1} \ \bar{1} \ Lq_2)$. 

When the head of the UTM moves to the left having written the symbol $\bar{1}$ in $Q_R$, then

$\bar{1} \ L \bar{1} \ (q_2 \ \bar{1} \ \bar{1} \ Lq_2)$,
$\bar{c} \ L \bar{c} \ (q_2 \ \bar{c} \ \bar{c} \ Lq_2)$,
$\bar{b} \ L \bar{b} \ (q_2 \ \bar{b} \ \bar{b} \ Lq_2)$. 

If the head of the UTM moves to the left in the code $P$, after writing the symbol $\bar{1}$ in $Q_R$ and meets the mark $b$, then the UTM writes the mark $c_2$ in $Q_R$:

$L \ b \ (b_2)R \ (q_2 \ \bar{b} \ b \ Rq_1)$,
$\bar{b} \ R \bar{b}_1 \ (q_1 \ \bar{b} \ b_1 \ Rq_1)$,
$\bar{1} \ R \bar{1}_1 \ (q_1 \ \bar{1} \ \bar{1}_1 \ Rq_1)$,
$\bar{c} \ R \bar{c} \ (q_1 \ \bar{c} \ \bar{c} \ Rq_1)$,
$R \ \bar{1} \ (c_2)L \ (q_1 \ \bar{1} \ c_2 \ Lq_1)$. 

When the head of the UTM moves to the left after writing the symbol $c_2$ in $Q_R$, then

$\bar{1}_1 \ L \bar{1}_1 \ (q_1 \ \bar{1}_1 \ \bar{1}_1 \ Lq_1)$,
$\bar{c} \ L \bar{c}_1 \ (q_1 \ \bar{c} \ \bar{c}_1 \ Lq_1)$,
$\bar{b}_1 \ L \bar{b}_1 \ (q_1 \ \bar{b}_1 \ \bar{b}_1 \ Lq_1)$. 

The UTM halts on the second stage when it locates the symbol $\bar{c}_1$ in the state $q_1$:

$(q_2 \ \bar{c}_1 \ c_2 \ Lq_1, \ q_1 \ \bar{c}_1-)$. 
Now there are two variants when the head of the UTM moves to the left and meets the pairs $1b_2$ or $bb_2$:

(a) $Lb_2(\bar{1}b_1)R$ ($q_1b_2b_3Lq_2$, $q_21\bar{1}Rq_2$, $q_2b_3\bar{1}Rq_2$) and the UTM restores the part $S$ of the tape and the second stage continues:

\[
\begin{array}{cc}
\bar{1}_1 R \bar{1}_1 & (q_2 \bar{1}_1 \bar{1}_1 Rq_2), \\
\bar{b}_1 R \bar{b}_1 & (q_2 \bar{b}_1 \bar{b}_1 Rq_2), \\
\bar{c}_1 R c_2 & (q_2 \bar{c}_1 c_2 Rq_2), \\
R c_2 (c)L & (q_2 c_2 c Lq_2), \\
\bar{1}_1 L 1 & (q_2 \bar{1}_1 1 Lq_2), \\
c_2 L c & (q_2 c_2 c Lq_2), \\
\bar{b}_1 L \bar{b} & (q_2 \bar{b}_1 \bar{b} Lq_2).
\end{array}
\]

Now the UTM can write the symbol $\bar{1}$ in $Q_R$.

(b) $Lb_2(bb)R$ ($q_1b_2b_3Lq_2$, $q_2b_2b_2Rq_1$, $q_1b_3\bar{b}_1Lq_2$, $q_2b_2bRq_1$, $q_1\bar{b}_1bRq_1$) and the third stage begins:

(iii) On the third stage of modelling the UTM restores the tape in $P$:

\[
\begin{array}{cc}
\bar{1}_1 R 1 & (q_1 \bar{1}_1 1 Rq_1), \\
\bar{b}_1 R b & (q_1 \bar{b}_1 b Rq_1).
\end{array}
\]

When the head of the UTM moves to the right and meets the mark $\bar{c}_1$, then this mark is changed to $\bar{c}_1$ ($q_1\bar{c}_1\bar{c}_1Rq_2$) and, then, the UTM restores the tape in $S$:

\[
\begin{array}{cc}
\bar{1}_1 R \bar{1}_1 & (q_2 \bar{1}_1 \bar{1}_1 Rq_2), \\
\bar{c}_1 R c_2 & (q_2 \bar{c}_1 c_2 Rq_2), \\
R c_2 (c)L & (q_2 c_2 c Lq_2), \\
\bar{1}_1 L 1 & (q_2 \bar{1}_1 1 Lq_2), \\
c_2 L c & (q_2 c_2 c Lq_2).
\end{array}
\]

When the head of the UTM moves to the left and meets the mark $\bar{c}_1$, both the third stage and the whole cycle of modelling are over. The UTM deletes the mark $\bar{c}_1$ and a new cycle of modelling begins ($q_2 \bar{c}_1 c_2 Lq_1$).

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References


